

NONNOETHERIAN COORDINATE RINGS WITH MULTIPLE POSITIVE DIMENSIONAL POINTS

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ABSTRACT. Given an affine algebraic variety X and a finite collection of non-intersecting positive dimensional algebraic sets $Y_i \subset X$, we construct a nonnoetherian coordinate ring whose variety coincides with X except that each Y_i is identified as a distinct positive dimensional closed point.

1. INTRODUCTION

Let S be an integral domain and a finitely generated algebra over an algebraically closed field k . Then for any maximal ideal $\mathfrak{n} \in \text{Max } S$, we have $S = k + \mathfrak{n}$. More generally, we will show that if I is a nonzero radical ideal of S , then

$$\dim S/I = 0$$

if and only if the ring

$$(1) \quad R = k + I$$

is noetherian (Corollary 1.3). Rings of the form (1) with $\dim S/I \geq 1$ comprise a basic class of examples in the study of nonnoetherian algebraic geometry [B2, B4]. Geometrically, the maximal spectrum $\text{Max } R$ coincides with the algebraic variety $\text{Max } S$, except that the zero locus $\mathcal{Z}(I) \subset \text{Max } S$ is identified as a single ‘smeared-out’ positive dimensional closed point of $\text{Max } R$.

Here we consider the question: *given a collection of pair-wise coprime ideals*

$$I_1, \dots, I_n \subset S,$$

is there a nonnoetherian ring R for which $\text{Max } R$ coincides with $\text{Max } S$, except that each $\mathcal{Z}(I_i)$ is identified as a distinct closed point of $\text{Max } R$? We show that this question has a positive answer, with R given by the intersection

$$R = \cap_i (k + I_i).$$

Recall that a depiction of a nonnoetherian domain R is a finitely generated k -algebra S that is as close to R as possible, in a suitable geometric sense (Definition

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2.1). For example, if R is depicted by S , then R and S have equal Krull dimension, and their maximal spectra are birationally equivalent [B1, Theorem 2.5]. Set

$$(2) \quad U_{S/R} := \{\mathfrak{n} \in \text{Max } S \mid R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}\}.$$

Our main theorem is the following.

Theorem 1.1. *(Propositions 3.3, 3.4, and Theorem 3.14.) Let X be an affine algebraic variety over k with coordinate ring S . Consider a collection of pair-wise non-intersecting algebraic sets Y_1, \dots, Y_n of X , where each ideal $I(Y_i)$ is proper, nonzero, and non-maximal. Then the maximal spectrum of the ring*

$$R := \cap_i (k + I(Y_i))$$

coincides with X except that each Y_i is identified as a distinct closed point. In particular, the locus $U_{S/R} \subset X$ is given by the intersection of the open sets Y_i^c ,

$$U_{S/R} = \cap_i Y_i^c.$$

Furthermore,

- (1) R is nonnoetherian if and only if there is some i for which $\dim Y_i \geq 1$.
- (2) R is depicted by S if and only if for each i , $\dim Y_i \geq 1$.

Example 1.2. Let $S = k[x, y]$, and consider the 3 lines in $\text{Max } S$,

$$(3) \quad \mathcal{Z}(x) = \{x = 0\}, \quad \mathcal{Z}(x - 1) = \{x = 1\}, \quad \mathcal{Z}(x - 2) = \{x = 2\}.$$

By Theorem 1.1, the coordinate ring

$$\begin{aligned} R &= (k + xS) \cap (k + (x - 1)S) \cap (k + (x - 2)S) \\ &= k[x] + x(x - 1)(x - 2)S \end{aligned}$$

is nonnoetherian and depicted by S . Furthermore, $\text{Max } R$ coincides with $\text{Max } S$ except that each of the 3 lines in (3) is identified as a 1-dimensional closed point.

For the special case $n = 1$, Theorem 1.1 implies the following.

Corollary 1.3. *Suppose I is a proper nonzero non-maximal radical ideal of S , and set $R = k + I$. Then the following are equivalent:*

- (1) $\dim S/I \geq 1$.
- (2) R is nonnoetherian.
- (3) R is depicted by S .

2. PRELIMINARY DEFINITIONS

Throughout, k is an algebraically closed field, and S is an integral domain and a finitely generated k -algebra. Let R be a (possibly nonnoetherian) subalgebra of S . Denote by $\text{Max } S$, $\text{Spec } S$, and $\dim S$ the maximal spectrum (or variety), prime spectrum (or affine scheme), and Krull dimension of S respectively; similarly for R . For a subset $I \subset S$, set $\mathcal{Z}(I) := \{\mathfrak{n} \in \text{Max } S \mid \mathfrak{n} \supseteq I\}$.

The following definitions were introduced in [B1] to study a class of noncommutative quiver algebras called dimer algebras [B2, B3, B4]. Recall the open subset $U_{S/R} \subset \text{Max } S$ defined in (2).

Definition 2.1. [B1, Definition 3.1]

- We say S is a *depiction* of R if the morphism

$$\iota_{S/R} : \text{Spec } S \rightarrow \text{Spec } R, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$

is surjective, and

$$U_{S/R} = \{\mathfrak{n} \in \text{Max } S \mid R_{\mathfrak{n} \cap R} \text{ is noetherian}\} \neq \emptyset.$$

- The *geometric height* of $\mathfrak{p} \in \text{Spec } R$ is the minimum

$$\text{ght}(\mathfrak{p}) := \min \left\{ \text{ht}_S(\mathfrak{q}) \mid \mathfrak{q} \in \iota_{S/R}^{-1}(\mathfrak{p}), S \text{ a depiction of } R \right\}.$$

The *geometric dimension* of \mathfrak{p} is¹

$$\text{gdim } \mathfrak{p} := \dim R - \text{ght}(\mathfrak{p}).$$

For brevity, we will often write ι for $\iota_{S/R}$.

3. PROOF OF MAIN THEOREM

Let I_1, \dots, I_n be a collection of proper nonzero non-maximal ideals of S such that for each $i \neq j$,

$$\mathcal{Z}(I_i) \cap \mathcal{Z}(I_j) = \emptyset.$$

Equivalently, I_i and I_j are coprime ideals, $I_i + I_j = S$. Unless stated otherwise, we denote by R the algebra

$$R := \cap_i (k + I_i).$$

Remark 3.1. If some I_j were a maximal ideal of S , then $k + I_j = S$; whence $R = \cap_{i \neq j} (k + I_i)$. The assumption that each I_i is nonzero and non-maximal implies that $\dim S \geq 1$.

Lemma 3.2. *Suppose $n \geq 2$. For each $1 \leq i \leq n$, there are elements $a, b \in R$ satisfying*

$$a \in I_i \setminus (\cup_{j \neq i} I_j), \quad b \in (\cap_{j \neq i} I_j) \setminus I_i,$$

and which sum to unity, $a + b = 1$.

Proof. Fix $1 \leq i \leq n$. By assumption,

$$\mathcal{Z}(1) = \emptyset = \mathcal{Z}(I_i) \cap (\cup_{j \neq i} \mathcal{Z}(I_j)) = \mathcal{Z}(I_i + \cap_{j \neq i} I_j).$$

Whence

$$1 \in I_i + \cap_{j \neq i} I_j.$$

¹Recall that if S is an integral domain and a finitely generated k -algebra, then for each $\mathfrak{q} \in \text{Spec } S$, we have $\dim S/\mathfrak{q} = \dim S - \text{ht}(\mathfrak{q})$.

Thus there is some $a \in I_i$ and $b \in \cap_{j \neq i} I_j$ such that $a + b = 1$. In particular,

$$a = 1 - b \in I_i \cap (\cap_{j \neq i} (k + I_j)) \subset R.$$

Furthermore, for each $j \neq i$,

$$a = 1 - b \in I_i \setminus I_j \quad \text{and} \quad b = 1 - a \in I_j \setminus I_i.$$

□

Proposition 3.3. *Each ideal $I_i \cap R$ is a distinct closed point of $\text{Spec } R$.*

Proof. Fix i . Then for each $a \in R \subseteq (k + I_i)$, there is some $\alpha_i \in k$ and $b_i \in I_i$ such that $a = \alpha_i + b_i$. In particular, there is an algebra epimorphism

$$R \rightarrow k, \quad a \mapsto \alpha_i,$$

with kernel $I_i \cap R$; whence an algebra isomorphism

$$R/(I_i \cap R) \cong k.$$

Furthermore, for each $j \neq i$ we have

$$I_j \cap R \neq I_i \cap R,$$

by Lemma 3.2. Therefore each $I_i \cap R$ is a distinct maximal ideal of R . □

Proposition 3.4. *The locus $U_{S/R}$ is given by*

$$U_{S/R} = (\cup_i \mathcal{Z}(I_i))^c.$$

Proof. (i) We first claim that $U_{S/R} \subseteq (\cup_i \mathcal{Z}(I_i))^c$. Indeed, let $\mathfrak{n} \in \cup_i \mathcal{Z}(I_i)$. Then \mathfrak{n} contains some I_i . By assumption, I_i is a non-maximal ideal of S . Thus there is another maximal ideal $\mathfrak{n}' \neq \mathfrak{n}$ of S which contains I_i . Whence

$$I_i \cap R \subseteq \mathfrak{n} \cap R \neq R \quad \text{and} \quad I_i \cap R \subseteq \mathfrak{n}' \cap R \neq R.$$

But $I_i \cap R$ is a maximal ideal of R by Proposition 3.3. Therefore

$$\mathfrak{n} \cap R = I_i \cap R = \mathfrak{n}' \cap R.$$

Now fix $c \in \mathfrak{n} \setminus \mathfrak{n}'$. Assume to the contrary that $c \in R_{\mathfrak{n} \cap R}$. Then there is some $a \in R$ and $b \in R \setminus (\mathfrak{n} \cap R)$ such that $c = \frac{a}{b}$. Whence

$$a = bc \in \mathfrak{n} \cap R = \mathfrak{n}' \cap R.$$

In particular,

$$bc \in \mathfrak{n}'$$

with $b, c \in S$. Therefore

$$(4) \quad b \in \mathfrak{n}',$$

since $c \notin \mathfrak{n}'$ and \mathfrak{n}' is a prime ideal of S . But $b \in R$ and

$$b \notin \mathfrak{n} \cap R = \mathfrak{n}' \cap R.$$

Whence $b \notin \mathfrak{n}'$, a contradiction to (4). Thus $c \in S_{\mathfrak{n}} \setminus R_{\mathfrak{n} \cap R}$. Therefore $\mathfrak{n} \in U_{S/R}^c$.

(ii) We now claim that $U_{S/R} \supseteq (\cup_i \mathcal{Z}(I_i))^c$.² Let $\mathfrak{n} \in (\cup_i \mathcal{Z}(I_i))^c$. Then for each i , $\mathfrak{n} \not\supseteq I_i$. In particular, for each i there is some $c_i \in I_i \setminus \mathfrak{n}$. Furthermore, since \mathfrak{n} is prime, we have

$$(5) \quad c := c_1 \cdots c_n \in (\cap_i I_i) \setminus \mathfrak{n}.$$

Now let $\frac{a}{b} \in S_{\mathfrak{n}}$, with $a \in S$ and $b \in S \setminus \mathfrak{n}$. Then by (5),

$$ac \in R \quad \text{and} \quad bc \in R \setminus (\mathfrak{n} \cap R).$$

Thus

$$\frac{a}{b} = \frac{ac}{bc} \in R_{\mathfrak{n} \cap R}.$$

Whence

$$S_{\mathfrak{n}} \subseteq R_{\mathfrak{n} \cap R} \subseteq S_{\mathfrak{n}}.$$

Therefore $R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}$. □

Lemma 3.5. *If J is a proper ideal of R and $\mathcal{Z}(J) \cap U_{S/R} = \emptyset$, then J is contained in some I_i .*

Proof. Suppose the hypotheses hold, and let $\mathfrak{n} \in \mathcal{Z}(J)$. Then $\mathfrak{n} \in U_{S/R}^c$. Whence $\mathfrak{n} \in \cup_j \mathcal{Z}(I_j)$ by Proposition 3.4. Thus \mathfrak{n} contains some I_i . In particular,

$$I_i \cap R \subseteq \mathfrak{n} \cap R \neq R.$$

Whence $I_i \cap R = \mathfrak{n} \cap R$ since $I_i \cap R \in \text{Max } R$ by Proposition 3.3. Therefore

$$J = J \cap R \subseteq \mathfrak{n} \cap R = I_i \cap R \subseteq I_i.$$

□

Lemma 3.6. *For each i ,*

$$(6) \quad R_{I_i \cap R} = (k + I_i)_{I_i}.$$

Proof. The lemma is trivial if $n = 1$, so suppose $n \geq 2$. Fix $1 \leq i \leq n$. By Lemma 3.2, there is some

$$c \in (\cap_{j \neq i} I_j) \cap R \setminus I_i.$$

Let $\frac{a}{b} \in (k + I_i)_{I_i}$, with $a \in k + I_i$ and $b \in (k + I_i) \setminus I_i$. Then

$$ac \in R \quad \text{and} \quad bc \in R \setminus (I_i \cap R),$$

where $bc \notin I_i$ since I_i is a maximal, hence prime, ideal of $k + I_i$. Whence

$$\frac{a}{b} = \frac{ac}{bc} \in R_{I_i \cap R}.$$

Thus

$$(k + I_i)_{I_i} \subseteq R_{I_i \cap R}.$$

Conversely,

$$R_{I_i \cap R} = (\cap_j (k + I_j))_{I_i \cap R} \subseteq \cap_j (k + I_j)_{I_i \cap (k + I_j)} \subseteq (k + I_i)_{I_i \cap (k + I_i)} = (k + I_i)_{I_i}.$$

²This claim was proven in the special case $n = 1$ in [B1, Proposition 2.8].

Therefore (6) holds. □

For the following, note that if $\mathfrak{n}_1, \dots, \mathfrak{n}_\ell$ are maximal ideals of S , then

$$I = \sqrt{\mathfrak{n}_1 \cdots \mathfrak{n}_\ell}$$

is a radical ideal of S satisfying $\dim S/I = 0$.

Lemma 3.7. *Suppose I is a radical ideal of S satisfying $\dim S/I = 0$. Then the ring $R = k + I$ is noetherian.*

Proof. Suppose R is nonnoetherian. We claim that

$$\dim S/I = \dim \mathcal{Z}(I) \stackrel{(i)}{=} \dim U_{S/R}^c \stackrel{(ii)}{\geq} 1.$$

Indeed, (i) holds since by Proposition 3.4,

$$\mathcal{Z}(I) = U_{S/R}^c.$$

To show (ii), recall [B1, Theorem 3.13.2]:³ if R is a nonnoetherian subalgebra of a finitely generated k -algebra S , and there is some $\mathfrak{m} \in \iota(U_{S/R}^c)$ satisfying $\sqrt{\mathfrak{m}S} = \mathfrak{m}$, then

$$\dim U_{S/R}^c \geq 1.$$

In our case, $R = k + I$ is nonnoetherian, $\sqrt{IS} = I$, and by Proposition 3.4,

$$I \in \iota(U_{S/R}^c).$$

Therefore (ii) holds. □

Proposition 3.8. *Suppose each I_i is a radical ideal of S .*

- (1) *If $\dim S/I_i = 0$ for each i , then R is noetherian.*
- (2) *If $\dim S/I_i = 0$, then the localization $R_{I_i \cap R}$ is noetherian.*

Proof. (1) Suppose $\dim S/I_i = 0$ for each i . Set

$$R^m := \cap_{i=1}^m (k + I_i).$$

We proceed by induction on m .

By Lemma 3.7, R^1 is noetherian. So suppose R^m is noetherian; we claim that R^{m+1} is noetherian.

Indeed, recall that a ring T is noetherian if there is a finite set of elements $a_1, \dots, a_m \in T$ such that $(a_1, \dots, a_m)T = T$, and each localization $T_{a_i} := T[a_i^{-1}]$ is noetherian (e.g., [H, Proposition III.3.2]).

By Lemma 3.2, R^{m+1} contains elements

$$(7) \quad a \in I_{m+1} \setminus (\cup_{i=1}^m I_i) \quad \text{and} \quad b \in (\cap_{i=1}^m I_i) \setminus I_{m+1}$$

³In the published version of [B1, Theorem 3.13.2], S is assumed to be a depiction of R , but this is not used in the proof of the theorem.

satisfying $a + b = 1$. In particular,

$$(a, b)R^{m+1} = R^{m+1}.$$

Furthermore, (7) implies

$$(8) \quad R_a^{m+1} = R_a^m \quad \text{and} \quad R_b^{m+1} = (k + I_{m+1})_b.$$

But R^m is noetherian by assumption, and $(k + I_{m+1})$ is noetherian by Lemma 3.7. Thus the localizations (8) are noetherian. Therefore R^{m+1} is noetherian, proving our claim.

(2) Now suppose $\dim S/I_i = 0$. Then the ring $k + I_i$ is noetherian by Lemma 3.7. Thus the localization $(k + I_i)_{I_i}$ is noetherian. But $R_{I_i \cap R} = (k + I_i)_{I_i}$ by Lemma 3.6. Therefore $R_{I_i \cap R}$ is noetherian. \square

Proposition 3.9. *Suppose I is a nonzero ideal of S satisfying $\dim S/I \geq 1$. Then the ring $R = k + I$ is nonnoetherian and I contains a strict infinite ascending chain of ideals of R .⁴*

Proof. Suppose $\dim S/I \geq 1$. Then I is a non-maximal ideal of S . Let x_1, \dots, x_n be a minimal generating set for S over k . Since I is non-maximal and k is algebraically closed, there is some $1 \leq j \leq n$ such that for each $\alpha \in k$,

$$x_j - \alpha \notin I,$$

by Hilbert's Nullstellensatz. Set $h := x_j$. Then for each $\alpha \in k$, $(I, h - \alpha)S$ is a proper ideal of S . Thus there is a maximal ideal $\mathfrak{n}_\alpha \in \text{Max } S$ such that

$$(I, h - \alpha) \subseteq \mathfrak{n}_\alpha.$$

Furthermore,

$$I \subseteq (I, h - \alpha) \cap R \subseteq \mathfrak{n}_\alpha \cap R \neq R.$$

Therefore, since I is a maximal ideal of R ,

$$(9) \quad \mathfrak{n}_\alpha \cap R = I.$$

By assumption, $I \neq 0$. Let $g \in I \setminus 0$, and consider the chain of ideals of R ,

$$0 \subset gR \subseteq (g, gh)R \subseteq (g, gh, gh^2)R \subseteq \dots \subseteq I.$$

We claim that each inclusion is proper. Indeed, assume to the contrary that there is some $\ell \geq 0$ and $r_0, \dots, r_\ell \in R$ such that

$$gh^{\ell+1} = \sum_{j=0}^{\ell} r_j gh^j.$$

⁴This proposition is erroneously claimed as a corollary to [B1, Theorem 3.13, published version]. [B1, Theorem 3.13] assumes that S is a depiction of R , but if R is noetherian, then S will not be a depiction of R . Indeed, in this case the only depiction of R will be itself [B1, Theorem 3.12], and $R \neq S$ if I is a non-maximal ideal of S .

Then since S is an integral domain,

$$h^{\ell+1} = \sum_{j=0}^{\ell} r_j h^j.$$

Whence

$$s := h^{\ell+1} - \sum_{j=1}^{\ell} r_j h^j = r_0 \in R.$$

But $h \in \mathfrak{n}_0$. Thus $s \in \mathfrak{n}_0$. Therefore $s \in \mathfrak{n}_0 \cap R = I$ by (9).

Since $R = k + I$, for each $0 \leq j \leq \ell$ there is some $\beta_j \in k$ and $t_j \in I$ such that $r_j = \beta_j + t_j$. Thus, since s and each $t_j h^j$ are in I , we have

$$(10) \quad t := h^{\ell+1} - \sum_{j=1}^{\ell} \beta_j h^j = s + \sum_{j=1}^{\ell} t_j h^j \in I.$$

The left-hand side implies that t is a polynomial in $k[h]$. Therefore t splits

$$t = h^{\ell+1} - \sum_{j=1}^{\ell} \beta_j h^j = (h - \alpha_0)(h - \alpha_1) \cdots (h - \alpha_{\ell}),$$

with $\alpha_0, \dots, \alpha_{\ell} \in k$, since k is algebraically closed. Furthermore, k is infinite, again since k is algebraically closed. Thus we may fix a scalar

$$\alpha' \in k \setminus \{\alpha_0, \dots, \alpha_{\ell}\}.$$

Then for each j ,

$$h - \alpha_j \notin \mathfrak{n}_{\alpha'}$$

since $h - \alpha' \in \mathfrak{n}_{\alpha'}$. Therefore t is not in $\mathfrak{n}_{\alpha'}$ since $\mathfrak{n}_{\alpha'}$ is prime. But by (10),

$$t \in I \subset (I, h - \alpha') \subseteq \mathfrak{n}_{\alpha'},$$

a contradiction. □

Proposition 3.10. *Suppose $\dim S/I_i \geq 1$ for some i . Then $R = \cap_i (k + I_i)$ is nonnoetherian.*

Proof. Suppose $\dim S/I_i \geq 1$. By Proposition 3.9, I_i contains a strict infinite ascending chain of ideals of $k + I_i$,

$$J_1 \subset J_2 \subset J_3 \subset \cdots \subset I_i.$$

(i) We claim that each J_{ℓ} is an R -module. Indeed, let $r \in R$. Then $r \in k + I_i$. Whence $J_{\ell} r \subseteq J_{\ell}$ since J_{ℓ} is an ideal of $k + I_i$, proving our claim.

(ii) Now let $a \in \cap_j I_j$. Then each $a J_{\ell}$ is in $\cap_j I_j \subset R$. Thus each $a J_{\ell}$ is an ideal of R by Claim (i).

Consider the chain of ideals of R ,

$$(11) \quad a J_1 \subseteq a J_2 \subseteq a J_3 \subseteq \cdots$$

Assume to the contrary that for some ℓ ,

$$aJ_\ell = aJ_{\ell+1}.$$

Then for each $b \in J_{\ell+1} \setminus J_\ell$, there is some $c \in J_\ell$ such that

$$ab = ac.$$

But S is an integral domain. Whence

$$b = c \in J_\ell,$$

a contradiction to our choice of b . Thus the chain (11) is strict. Therefore R is nonnoetherian. \square

We recall the following elementary facts.

Lemma 3.11. *Let R be an integral domain, and let $\mathfrak{p}, \mathfrak{m} \in \text{Spec } R$ be ideals satisfying $\mathfrak{p} \subseteq \mathfrak{m}$. Then⁵*

- (1) $\mathfrak{p}R_{\mathfrak{m}} \cap R = \mathfrak{p}$.
- (2) $\mathfrak{p}R_{\mathfrak{m}} \in \text{Spec } R_{\mathfrak{m}}$.

Again let $R = \cap_i (k + I_i)$.

Lemma 3.12. *If $\mathfrak{p} \in \text{Spec } R$ and $\mathfrak{p} \subseteq I_i$ for some i , then*

$$\mathfrak{p}S \cap R = \mathfrak{p}.$$

Proof. Suppose the hypotheses hold. Let $ab \in \mathfrak{p}S \cap R$, with $a \in \mathfrak{p}$ and $b \in S$. We claim that $ab \in \mathfrak{p}$. Indeed, by Lemma 3.2 there is some

$$c \in (\cap_{j \neq i} I_j) \cap R \setminus I_i.$$

⁵We prove Lemma 3.11 for completeness.

(1) It suffices to show that $\mathfrak{p}R_{\mathfrak{m}} \cap R \subseteq \mathfrak{p}$. Let $\frac{a}{b} \in \mathfrak{p}R_{\mathfrak{m}}$, with $a \in \mathfrak{p}$ and $b \in R \setminus \mathfrak{m}$. Then

$$b \cdot \frac{a}{b} = a \in \mathfrak{p}.$$

Thus, since $b, \frac{a}{b} \in R$ and \mathfrak{p} is prime in R , we have $b \in \mathfrak{p}$ or $\frac{a}{b} \in \mathfrak{p}$. But $b \notin \mathfrak{p}$ since $b \notin \mathfrak{m}$ and $\mathfrak{p} \subseteq \mathfrak{m}$. Therefore $\frac{a}{b} \in \mathfrak{p}$.

(2) Let $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in R_{\mathfrak{m}}$, with $a_1, a_2 \in R$ and $b_1, b_2 \in R \setminus \mathfrak{m}$. Suppose

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \in \mathfrak{p}R_{\mathfrak{m}}.$$

We claim that $\frac{a_1}{b_1}$ or $\frac{a_2}{b_2}$ is in $\mathfrak{p}R_{\mathfrak{m}}$. Indeed, there is some $c \in \mathfrak{p}$ and $d \in R \setminus \mathfrak{m}$ such that

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{c}{d}.$$

Whence

$$a_1 a_2 d = b_1 b_2 c \in \mathfrak{p}.$$

Now $d \notin \mathfrak{p}$ since $d \notin \mathfrak{m}$ and $\mathfrak{p} \subseteq \mathfrak{m}$. Thus $a_1 a_2 \in \mathfrak{p}$ since \mathfrak{p} is prime in R . In particular, $a_1 \in \mathfrak{p}$ or $a_2 \in \mathfrak{p}$; say $a_1 \in \mathfrak{p}$. Then $\frac{a_1}{b_1} \in \mathfrak{p}R_{\mathfrak{m}}$, proving our claim.

Then $ac \in \cap_j I_j$ since $a \in \mathfrak{p} \subseteq I_i$. Thus for any $s \in S$,

$$acs \in \cap_j I_j \subset R.$$

In particular,

$$acb^2 \in R.$$

Thus, since $a \in \mathfrak{p}$,

$$(ab)^2 \cdot c = a \cdot (acb^2) \in \mathfrak{p}.$$

But $c \in R \setminus \mathfrak{p}$ and $(ab)^2 \in R$. Thus $(ab)^2 \in \mathfrak{p}$ since \mathfrak{p} is prime in R . Therefore $ab \in \mathfrak{p}$, again since \mathfrak{p} is prime in R . \square

Proposition 3.13. *The morphism*

$$\iota : \operatorname{Spec} S \rightarrow \operatorname{Spec} R, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$

is surjective.

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$. We claim that there is some $\mathfrak{q} \in \operatorname{Spec} S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.

(i) First suppose $\mathcal{Z}(\mathfrak{p}) \cap U_{S/R} = \emptyset$. Then there is some i for which $\mathfrak{p} \subseteq I_i$ by Lemma 3.5. Set

$$\mathfrak{t} := \mathfrak{p}(k + I_i)_{I_i} \cap (k + I_i).$$

Recall that $I_i \cap R \in \operatorname{Spec} R$ by Proposition 3.3.

(i.a) We first claim that $\mathfrak{p} = \mathfrak{t} \cap R$. Indeed,

$$\mathfrak{p} \stackrel{(i)}{=} \mathfrak{p}R_{I_i \cap R} \cap R \stackrel{(ii)}{=} \mathfrak{p}(k + I_i)_{I_i} \cap R = \mathfrak{p}(k + I_i)_{I_i} \cap (k + I_i) \cap R = \mathfrak{t} \cap R,$$

where (i) holds by Lemma 3.11.1, and (ii) holds by Lemma 3.6.

(i.b) We claim that

$$\mathfrak{t} \in \operatorname{Spec}(k + I_i) \quad \text{and} \quad \mathfrak{t} \subseteq I_i.$$

By Lemma 3.11.2,

$$\mathfrak{p}R_{I_i \cap R} \in \operatorname{Spec} R_{I_i \cap R}.$$

Thus by Lemma 3.6,

$$\mathfrak{p}(k + I_i)_{I_i} \in \operatorname{Spec}(k + I_i)_{I_i}.$$

Therefore $\mathfrak{t} \in \operatorname{Spec}(k + I_i)$.

Furthermore,

$$\mathfrak{t} = \mathfrak{p}(k + I_i)_{I_i} \cap (k + I_i) \subseteq I_i(k + I_i)_{I_i} \cap (k + I_i) \stackrel{(i)}{=} I_i,$$

where (i) holds by Lemma 3.11.1 since $I_i \in \operatorname{Spec}(k + I_i)$.

(i.c) We claim that

$$\mathfrak{p} = \sqrt[s]{\mathfrak{t}S} \cap R.$$

Indeed,

$$\mathfrak{p} \stackrel{(i)}{=} \mathfrak{t} \cap R \subseteq \sqrt[s]{\mathfrak{t}S} \cap R \subseteq \sqrt[r]{\mathfrak{t}S \cap R} = \sqrt[r]{\mathfrak{t}S \cap (k + I_i) \cap R} \stackrel{(ii)}{=} \sqrt[r]{\mathfrak{t} \cap R} = \sqrt[r]{\mathfrak{p}} = \mathfrak{p},$$

where (I) holds by Claim (i.a), and (II) holds by Claim (i.b) together with Lemma 3.12.

(i.d) Since S is noetherian, the Lasker-Noether theorem implies that there are ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_m \in \text{Spec } S$, minimal over $\sqrt[t]{S}$, such that

$$\sqrt[t]{S} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_m.$$

Thus

$$(12) \quad \mathfrak{p} \stackrel{(I)}{=} \sqrt[t]{S} \cap R = (\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_m) \cap R = (\mathfrak{q}_1 \cap R) \cap \dots \cap (\mathfrak{q}_m \cap R),$$

where (I) holds by Claim (i.c). Furthermore, each $\mathfrak{q}_j \cap R$ is a prime ideal of R since $\mathfrak{q}_j \in \text{Spec } S$ and $R \subset S$ (e.g., [B1, Lemma 2.1]).

Assume to the contrary that for each $1 \leq j \leq m$,

$$\mathfrak{q}_j \cap R \neq \mathfrak{p}.$$

Then for each j there is some

$$a_j \in (\mathfrak{q}_j \cap R) \setminus \mathfrak{p}.$$

Whence

$$a_1 \cdots a_m \in \bigcap_j (\mathfrak{q}_j \cap R) \stackrel{(I)}{=} \mathfrak{p},$$

where (I) holds by (12). But \mathfrak{p} is prime in R , a contradiction. Thus there is some j for which

$$\mathfrak{q}_j \cap R = \mathfrak{p}.$$

Our desired ideal is therefore $\mathfrak{q} := \mathfrak{q}_j \in \text{Spec } S$.

(ii) Now suppose $\mathcal{Z}(\mathfrak{p}) \cap U_{S/R} \neq \emptyset$; say $\mathfrak{n} \in \mathcal{Z}(\mathfrak{p}) \cap U_{S/R}$. Set

$$\mathfrak{q} := \mathfrak{p}S_{\mathfrak{n}} \cap S.$$

We claim that

$$\mathfrak{q} \cap R = \mathfrak{p} \quad \text{and} \quad \mathfrak{q} \in \text{Spec } S.$$

Indeed,

$$\mathfrak{p} \stackrel{(I)}{=} \mathfrak{p}R_{\mathfrak{n} \cap R} \cap R \stackrel{(II)}{=} \mathfrak{p}S_{\mathfrak{n}} \cap R = \mathfrak{p}S_{\mathfrak{n}} \cap S \cap R = \mathfrak{q} \cap R,$$

where (I) holds by Lemma 3.11.1, and (II) holds since $\mathfrak{n} \in U_{S/R}$.

Furthermore, since $\mathfrak{p} \in \text{Spec } R$, we have $\mathfrak{p}R_{\mathfrak{n} \cap R} \in \text{Spec}(R_{\mathfrak{n} \cap R})$ by Lemma 3.11.2. Whence $\mathfrak{p}S_{\mathfrak{n}} \in \text{Spec } S_{\mathfrak{n}}$ since $\mathfrak{n} \in U_{S/R}$. Therefore $\mathfrak{q} = \mathfrak{p}S_{\mathfrak{n}} \cap S \in \text{Spec } S$. \square

Theorem 3.14. *Let I_1, \dots, I_n be a set of proper nonzero non-maximal radical ideals of S which are pair-wise coprime, and set $R := \bigcap_i (k + I_i)$. Then*

- (1) *R is nonnoetherian if and only if there is some i for which $\dim S/I_i \geq 1$.*
- (2) *R is depicted by S if and only if for each i , $\dim S/I_i \geq 1$.*

Proof. (1): Holds by Propositions 3.8.1 and 3.10.

(2): The morphism $\iota : \operatorname{Spec} S \rightarrow \operatorname{Spec} R$ is surjective by Proposition 3.13. Furthermore, $U_{S/R}$ is nonempty since $U_{S/R} = (\cup_i \mathcal{Z}(I_i))^c$ is an open dense subset of $\operatorname{Max} S$, by Proposition 3.4. It thus suffices to show that

$$(13) \quad U_{S/R}^c = \cup_i \mathcal{Z}(I_i) \subseteq \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n} \cap R} \text{ is nonnoetherian}\},$$

where the inclusion holds iff $\dim S/I_i \geq 1$ for each i .

Indeed, suppose $\mathfrak{n} \in \cup_i \mathcal{Z}(I_i)$. Then \mathfrak{n} contains some I_j . Whence $\mathfrak{n} \cap R = I_j \cap R$ by Proposition 3.3. Thus by Lemma 3.6,

$$R_{\mathfrak{n} \cap R} = R_{I_j \cap R} = (k + I_j)_{I_j}.$$

- First suppose $\dim S/I_j = 0$. Then $R_{\mathfrak{n} \cap R} = R_{I_j \cap R}$ is noetherian by Proposition 3.8.2. Therefore the inclusion in (13) does not hold.

- Now suppose $\dim S/I_j \geq 1$. Then I_j contains a strict infinite ascending chain of ideals of $k + I_j$, by Proposition 3.9. Therefore the localization $R_{\mathfrak{n} \cap R} = (k + I_j)_{I_j}$ is nonnoetherian. In particular, if $\dim S/I_i \geq 1$ for each i , then the inclusion in (13) holds. \square

Corollary 3.15. *If $\dim S/I_i \geq 1$ for each i , then each of the closed points $I_i \cap R$ of $\operatorname{Spec} R$ has positive geometric dimension.*

Proof. By Theorem 3.14, S is a depiction of R . Therefore for each i ,

$$\operatorname{gdim}(I_i \cap R) \geq \dim S/I_i \geq 1.$$

\square

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